Reduced fidelity susceptibility and its finite-size scaling behaviors in the Lipkin-Meshkov-Glick model

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We derive a general formula for the reduced fidelity susceptibility when the reduced density matrix is 2×2 block diagonal. By using this result and a continuous unitary transformation, we study finite-size scaling of the reduced fidelity susceptibility in the Lipkin-Meshkov-Glick model. It is found that it can well characterize quantum phase transitions, that is, we can extract information about quantum phase transitions from only the fidelity of a subsystem, which is of practical use in experiments.

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I. INTRODUCTION

During the past few years, some important concepts in quantum-information theory [1] have been introduced to characterize quantum phase transitions (QPTs) [2]. For example, entanglement, which is one of the central concepts in quantum-information theory, has been investigated extensively in QPTs in various models, like the Ising model [3–6] and Lipkin-Meshkov-Glick (LMG) model [7–10]. Recently, fidelity, another important quantum-information concept, has also been applied in characterizing QPTs. When a system undergoes QPTs, the ground state changes dramatically. Since the definition of fidelity is mathematically the overlap between two states, the introduction of fidelity in QPTs is natural [11–34]. However, in the study of QPTs, the fidelity depends computationally on an arbitrary yet finite small change of the controlling parameter. To cancel the arbitrariness, Zanardi *et al.* introduced the Riemannian metric tensor [16,17], while You *et al.* suggested the fidelity susceptibility [18]. The fidelity susceptibility then becomes an effective tool to study critical properties [16,21] in many-body systems.

So far, the most extensively studied fidelity in QPTs is the global ground state fidelity, which reflects the change of the global system. We put forward an issue about the fidelity of the subsystem when the global system undergoes a QPT. We call this kind of fidelity the reduced fidelity. In general, a subsystem, which is described by a reduced density matrix (RDM), is in a mixed state. Therefore, we introduce a general fidelity, i.e., the Uhlmann fidelilty, defined as [35]

$$F = \mathrm{tr}\sqrt{\rho^{1/2}\tilde{\rho}\rho^{1/2}}, \quad (1)$$

where ρ and $\tilde{\rho}$ are two different states, regardless of whether they are pure or mixed. The concept of the reduced fidelity was considered in Refs. [36,37], in which the fidelity is defined as

$$F = \mathrm{tr}(\rho\tilde{\rho}). \quad (2)$$

One of the two matrices is required to be a pure state to make the fidelity agree with the Uhlmann fidelity. In the rest of the paper we will consider the Uhlmann fidelity, given by Eq. (1), only. The investigation of reduced fidelity in QPTs is presented in [38,39], in which it is called the partial fidelity. In [40,41], the authors studied the reduced fidelity in renormalization group flows, and compared the critical behaviors between the reduced and the global fidelities. In our paper, we consider a two-body subsystem and introduce the reduced fidelity susceptibility (RFS). Moreover, we derive a general formula for the RFS under the condition that the RDM is block diagonal in 2×2 matrices. Then we study the two-spin RFS of the LMG model and find that its scaling exponent is different from that of the global one [24].

This paper is organized as follows. In Sec. II, we give a general formula for the RFS when the density matrix is block diagonal in 2×2 matrices. Then in Sec. III, we introduce the LMG model. In the thermodynamic limit, the RFS is divergent as $(1-h)^{-1}$ in the broken phase $(0\leqslant h<1)$, where h is the effective transverse magnetic field. However, it becomes zero in the entire symmetry phase $(h>1)$. In the finite-size situation, it is not suitable to use the RFS for the isotropic case in the symmetry phase due to energy degeneracy. For the anisotropic case, by using the continuous unitary transformation (CUT) [42–44], we find that the maximum of χ vs h diverges as $N^{2/3}$ for an N-spin system.

II. REDUCED FIDELITY SUSCEPTIBILITY

In general cases, the analytical calculation of the mixed state fidelity (1) is very hard; it involves exact diagonalization, which is usually performed numerically. Here we consider that the RDM ρ is block diagonal in a certain basis for any given parameters,

$$\rho = \bigoplus_{i=1}^{n} \varrho_i, \quad (3)$$

where ϱ_i is a 2×2 semipositive definite Hermitian matrix, and $2n$ is the dimension of ρ. The block-diagonal form is in general ensured by some symmetries of the system, and thus is independent of the parameters. In fact, this situation is common in a broad class of systems with special symmetries

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[38]. We consider $\rho \equiv \rho(h)$ and $\tilde{\rho} \equiv \rho(h+\delta)$, where h is a parameter in the Hamiltonian and δ is a small change.

The fidelity susceptibility is defined as [12,18] $\chi = \lim_{\delta\to 0}(-2\ln F)/\delta^2$, and approximately we have

$$F \simeq 1 - \chi\delta^2/2. \tag{4}$$

As ρ is block diagonal, the fidelity is written as

$$F = \sum_{i=1}^{n} F_i, \tag{5}$$

where $F_i = \mathrm{tr}\sqrt{\varrho_i^{1/2}\tilde{\varrho}_i\varrho_i^{1/2}}$ is the "fidelity" for the ith block. According to Eq. (4), we have

$$F_i \simeq \mathrm{tr}\varrho_i - \frac{\chi_i}{2}\delta^2, \tag{6}$$

where χ_i corresponds to the "susceptibility" of the ith block and $\chi = \Sigma_{i=1}^{n}\chi_i$.

As ϱ_i is a 2×2 matrix, we introduce the useful formula

$$\mathrm{tr}\sqrt{A^{1/2}BA^{1/2}} = \sqrt{\mathrm{tr}(AB) + 2\sqrt{\det(AB)}}, \tag{7}$$

where A and B are arbitrary 2×2 matrices of a certain argument h. This formula helps us avoid the computation of eigenvalues of ρ. As the RDM is semipositive definite, we restrict A and B to be the same. If $A=B$, it becomes

$$\mathrm{tr}(A^2) = (\mathrm{tr}A)^2 - 2\det A. \tag{8}$$

Taking derivatives of the above equation with respect to h, we get

$$\mathrm{tr}(AA') = \mathrm{tr}A\,\mathrm{tr}A' - \partial_h(\det A), \tag{9}$$

$$\mathrm{tr}(AA'') = \mathrm{tr}A\,\mathrm{tr}A'' - \partial_h^2(\det A) + 2\det A', \tag{10}$$

where $A' \equiv \partial_h A$, $A'' \equiv \partial_h^2 A$, and $\partial_h \mathrm{tr}(A) = \mathrm{tr}(A')$. Now we have

$$F_i = \sqrt{\mathrm{tr}(\varrho_i\tilde{\varrho}_i) + 2\sqrt{\det(\varrho_i\tilde{\varrho}_i)}}. \tag{11}$$

To obtain the susceptibility, we expand the fidelity with respect to δ; by using $\tilde{\varrho}_i \simeq \varrho_i(h) + \varrho_i'(h)\delta + \delta^2\varrho_i''(h)/2 + O(\delta^3)$, we get

$$\mathrm{tr}(\varrho_i\tilde{\varrho}_i) \simeq \mathrm{tr}(\varrho_i^2) + \mathrm{tr}(\varrho_i\varrho_i')\delta + \frac{\delta^2}{2}\mathrm{tr}(\varrho_i\varrho_i''),$$

$$\det\tilde{\varrho}_i \simeq \det\varrho_i + \partial_h(\det\varrho_i)\delta + \frac{\delta^2}{2}\partial_h^2(\det\varrho_i). \tag{12}$$

As we have used a series expansion of $\tilde{\rho}$ at h, it is necessary to know some properties of ρ in the vicinity of h. In this study, we emphasize that the form of ρ should stay the same as h changes, i.e., both ρ and $\tilde{\rho}$ are block diagonal in 2×2 matrices. This property of ρ is in general due to the symmetries of the Hamiltonian. However, as the parameter changes, the determinant and the trace of each block change as well. In the following, we discuss the calculation of fidelity in three cases, classified by the determinant and trace of ϱ_i, which are restricted in the region [0,1], because ϱ_i is a diagonal block of the density matrix ρ.

(i) $\det\varrho_i \neq 0$, $\mathrm{tr}\,\varrho_i \neq 0$. In this case the square root of $\det(\varrho_i\tilde{\varrho}_i)$ is expanded as

$$\sqrt{\det(\varrho_i\tilde{\varrho}_i)} \simeq \det\varrho_i + \frac{\delta}{2}\partial_h\det\varrho_i + \frac{\delta^2}{4}\left(\partial_h^2\det\varrho_i - \frac{(\partial_h\det\varrho_i)^2}{2\det\varrho_i}\right). \tag{13}$$

Taking the above expression into Eq. (11) and with the help of Eqs. (8)–(10), we obtain

$$F_i \simeq \mathrm{tr}\varrho_i + \frac{\delta}{2}\mathrm{tr}\varrho_i' + \frac{\delta^2}{4}\mathrm{tr}\varrho_i'' + \frac{\delta^2}{8\mathrm{tr}\varrho_i}\left(4\det\varrho_i' - (\mathrm{tr}\varrho_i')^2 - \frac{[\partial_h\det(\varrho_i)]^2}{\det(\varrho_i)}\right). \tag{14}$$

(ii) $\det\varrho_i = 0$, $\mathrm{tr}\,\varrho_i \neq 0$. This indicates $\det(\varrho_i\tilde{\varrho}_i)=0$ and $F_i = \sqrt{\mathrm{tr}(\varrho_i\tilde{\varrho}_i)}$. It is emphasized that ϱ_i is rank 1, but $\tilde{\varrho}_i$, in general, is not. Since the lower bound of $\det\varrho_i$ is zero, we have $\partial_h\det\varrho_i = 0$ and $\partial_h^2\det\varrho_i > 0$. Thus we have

$$\mathrm{tr}(\varrho_i\tilde{\varrho}_i) = (\mathrm{tr}\varrho_i)^2 + \mathrm{tr}\varrho_i\mathrm{tr}\varrho_i'\delta + \frac{\delta^2}{2}[\mathrm{tr}\varrho_i\mathrm{tr}\varrho_i'' - \partial_h^2(\det\varrho_i) + 2\det\varrho_i'] \tag{15}$$

and

$$F_i \simeq \mathrm{tr}\varrho_i + \frac{\delta}{2}\mathrm{tr}\varrho_i' + \frac{\delta^2}{4}\mathrm{tr}\varrho_i'' + \frac{\delta^2}{8\mathrm{tr}\varrho_i}[4\det\varrho_i' - (\mathrm{tr}\varrho_i')^2 - 2\partial_h^2(\det\varrho_i)]. \tag{16}$$

(iii) $\mathrm{tr}\,\varrho_i = 0$. As ϱ_i is Hermitian, it is equivalent to a zero matrix. Then $\mathrm{tr}(\varrho_i\tilde{\varrho}_i) = \sqrt{\det(\varrho_i\tilde{\varrho}_i)} = 0$, and $F_i = 0$.

Finally, we get the susceptibility for block ϱ_i:

$$\chi_i = \begin{cases} \dfrac{1}{4\mathrm{tr}\varrho_i}\left((\mathrm{tr}\varrho_i')^2 - 4\det\varrho_i' + \dfrac{(\partial_h\det\varrho_i)^2}{\det\varrho_i}\right) & \text{for } \mathrm{tr}\varrho_i \neq 0,\ \det\varrho_i \neq 0, \\ \dfrac{1}{4\mathrm{tr}\varrho_i}[(\mathrm{tr}\varrho_i')^2 - 4\det\varrho_i' + 2\partial_h^2(\det\varrho_i)] & \text{for } \mathrm{tr}\varrho_i \neq 0,\ \det\varrho_i = 0, \\ 0 & \text{for } \mathrm{tr}\varrho_i = 0, \end{cases} \tag{17}$$

where the terms $\delta \operatorname{tr} \varrho_i'/2$ and $\delta^2 \operatorname{tr} \varrho_i''/4$ in Eqs. (14) and (16) are canceled in the final expression of the fidelity, because $\operatorname{tr}(\rho) \equiv 1$ and $\operatorname{tr}(\rho') = \operatorname{tr}(\rho'') = 0$. Additionally, if ρ is diagonal in a certain basis for any given h, the susceptibility is obtained readily as

$$\chi = \sum_i^N \frac{(\lambda_i')^2}{4\lambda_i}, \tag{18}$$

where the λ_i's are the nonzero diagonal terms and N is the dimension of ρ.

III. LMG MODEL AND SCALING EXPONENTS OF THE RFS

A. LMG model and RFS

The LMG model was introduced in nuclear physics to describe mutually interacting spin-1/2 particles, embedded in a transverse magnetic field. The Hamiltonian reads

$$\begin{aligned} H &= -\frac{\lambda}{N}\sum_{i<j} (\sigma_x^i \sigma_x^j + \gamma \sigma_y^i \sigma_y^j) - h \sum_i \sigma_z^i \\ &= -\frac{2\lambda}{N}(S_x^2 + \gamma S_y^2) - 2hS_z + \frac{\lambda}{2}(1+\gamma), \end{aligned} \tag{19}$$

where σ_α $(\alpha = x,y,z)$ are the Pauli matrices and $S_\alpha = \Sigma_i \sigma_\alpha^i/2$ is the collective spin operator. N is the total spin number and the prefactor $1/N$ ensures finite energy per spin in the thermodynamic limit. $|\gamma| \leqslant 1$ is an anisotropy parameter; λ and h are parameters giving the spin-spin interaction strength and effective magnetic field, respectively. Here, we focus on the ferromagnetic case ($\lambda > 0$), and without loss of generality, we set $\lambda = 1$. We take $h \geqslant 0$ as the spectrum is invariant under the transformation $h \leftrightarrow -h$. In addition, we consider only the maximum spin sector $S = N/2$ in which the lowest-energy state lies. The ground-state properties can be easily studied in the thermodynamic limit by using a mean-field analysis. However, for finite-size case, the scaling of the spin expectation values has been studied by a $1/N$ expansion in the Holstein-Primakoff single-boson representation [45] and by the CUT [46,47]. The critical behavior of the global fidelity susceptibility of this model is studied in [24], in which the divergent form of the susceptibility is $1/(1-h)^2$ in symmetric phase and $1/\sqrt{(1-h)}$ in broken phase, and the finite-size scaling exponent is 1.33.

Now we consider a two-spin RDM under the ground state of the LMG model [48],

$$\rho_{ij} = \begin{pmatrix} v_+ & 0 & 0 & u \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & v_- \end{pmatrix}, \tag{20}$$

in the standard basis $\{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of σ_z with eigenvalues 1 and -1, respectively. The nonzero matrix elements read

$$v_\pm = \frac{N^2 - 2N + 4\langle S_z^2\rangle \pm 4\langle S_z\rangle(N-1)}{4N(N-1)},$$

$$y = \frac{N^2 - 4\langle S_z^2\rangle}{4N(N-1)}, \quad u = \frac{\langle S_x^2 - S_y^2\rangle}{N(N-1)}. \tag{21}$$

The zero elements of ρ_{ij} result from the fact that the total spin and the parity are conserved quantities, i.e.,

$$[H, S^2] = \left[H, \prod_{i=1}^N \sigma_{iz} \right] = 0. \tag{22}$$

It is noticed that ρ_{ij} is actually block diagonal in the rearranged basis $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$, and the two blocks are

$$\varrho_1 = \begin{pmatrix} v_+ & u \\ u & v_- \end{pmatrix}, \quad \varrho_2 = \begin{pmatrix} y & y \\ y & y \end{pmatrix}. \tag{23}$$

With the help of Eq. (17), we can give the RFS explicitly:

$$\begin{aligned} \chi = \frac{y'^2}{2y} + \frac{1}{4(v_+ + v_-)}\Bigg(& (v_+' - v_-')^2 + 4u'^2 \\ & + \frac{(v_+' v_- + v_+ v_-' - 2u'u)^2}{(v_+ v_- - u^2)} \Bigg); \end{aligned} \tag{24}$$

here we consider the case that $\det \varrho_1 \neq 0$, and the following computations are based on it. From the above formula, we see that the critical property of the RFS is determined by the elements of the density matrix, which consist of spin expectation values (21) and their first-order derivatives. The detailed calculation of these spin expectation values is presented in [46,47]. For the isotropic case ($\gamma = 1$), the Hamiltonian is diagonalized analytically. For anisotropic case ($\gamma \neq 1$), the exact spin expectation values are obtained by a mean-field approximation in the thermodynamic limit, and the scaling exponents of the spin expectation values are obtained by using the CUT method for finite N.

B. The thermodynamic limit

In the thermodynamic limit, the LMG model undergoes a second-order symmetry-breaking phase transition in the ferromagnetic regime [6]. For a strong magnetic field the system is in the symmetry phase, where the ground state is unique and polarized in the direction of the magnetic field. As the magnetic field is decreased below a critical value $h_c = 1$, the system enters the broken phase, where the ground state becomes doubly degenerate, thus breaking the parity symmetry.

In the following, we use a semiclassical approach to determine the phase diagram of the LMG model. This approach is exact in the thermodynamic limit for all γ and relies on a mean-field (variational) wave function

$$|\psi(\theta,\phi)\rangle = \bigoplus_{l=1}^N \left(\cos\frac{\theta}{2} e^{-i\phi/2} |\uparrow\rangle_l + \sin\frac{\theta}{2} e^{i\phi/2} |\downarrow\rangle_l \right), \tag{25}$$

which is a coherent spin state such that

$$\langle \mathbf{S} \rangle = \frac{N}{2}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{26}$$

The ground state is thus determined by minimizing the energy

$$\langle H\rangle = -\frac{(N-1)}{2}\sin^2\theta(\cos^2\phi+\gamma\sin^2\phi) - hN\cos\theta \tag{27}$$

with respect to θ and φ, leading to a distinction between the following two phases.

(i) $h\geq 1$ (symmetric phase). The ground state is unique and fully polarized in the magnetic field direction ($\theta_0=0$) for all γ.

(ii) $0\leq h<1$ (broken phase). For $\gamma\neq 1$, the ground state is twofold degenerate ($\theta_0=\arccos h$ and $\varphi_0=0$ or π). In the isotropic case $\gamma=1$, $\langle H\rangle$ does not depend on φ so that the ground state is infinitely degenerate.

Then the spin expectation values can be easily derived. For $h\geq 1$ we have

$$\lim_{N\to\infty}\frac{2\langle S_z\rangle}{N}=1,\quad \lim_{N\to\infty}\frac{4\langle S_z^2\rangle}{N^2}=1$$

$$\lim_{N\to\infty}\frac{4\langle S_x^2\rangle}{N^2}=\lim_{N\to\infty}\frac{4\langle S_y^2\rangle}{N^2}=0, \tag{28}$$

and for $0\leq h<1$,

$$\lim_{N\to\infty}\frac{2\langle S_z\rangle}{N}=h,\quad \lim_{N\to\infty}\frac{4\langle S_z^2\rangle}{N^2}=h^2,$$

$$\lim_{N\to\infty}\frac{4\langle S_x^2\rangle}{N^2}=\lim_{N\to\infty}\frac{4\langle S_y^2\rangle}{N^2}=\frac{1-h}{2}. \tag{29}$$

Now with Eq. (24), we can compute the RFS,

$$\chi=\begin{cases}0 & \text{for } h\geq 1,\\ \dfrac{1}{2(1-h^2)} & \text{for } 0\leq h<1.\end{cases} \tag{30}$$

The critical behavior is different from the global fidelity susceptibility studied in [24], in which the divergent form of the susceptibility is $1/(1-h)^2$ in symmetric phase and $1/\sqrt{(1-h)}$ in broken phase.

C. Finite-size scaling

For a finite-size system, we begin with the isotropic case, $\gamma=1$. The Hamiltonian is diagonal in the standard eigenbasis $\{|S,M\rangle\}$ of $\mathbf{S}^2$ and S_z. For $S=N/2$ the energy eigenvalue is

$$E(M,h)=\frac{2}{N}\left(M-\frac{hN}{2}\right)^2-\frac{N}{2}(1+h^2), \tag{31}$$

and the ground state $|S,M_0\rangle$ is readily obtained when

$$M_0=\begin{cases}N/2 & \text{for } h\geq 1,\\ N/2-R[N(1-h)/2] & \text{for } 0\leq h<1,\end{cases} \tag{32}$$

where $R(x)\equiv\text{round}(x)$ gives the nearest integer of x. Then one can see that level crossings exist at $h=h_j$, where $h_j=1-(2j+1)/N$, between the two states $|N/2,N/2-j\rangle$ and $|N/2,N/2-j-1\rangle$. As M_0 is not continuous in $0\leq h<1$, the spin expectation values are the same. Thus according to Eq. (24) there is no susceptibility in this case.

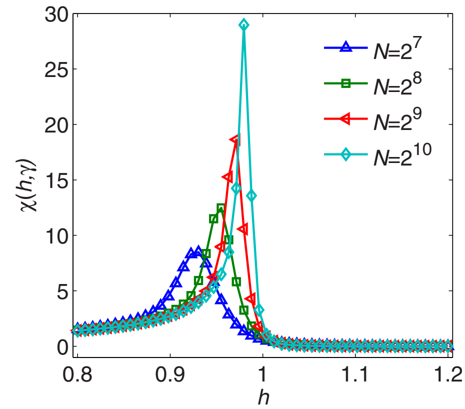

FIG. 1. (Color online) Fidelity susceptibility χ as a function of h with various system sizes $N=2^7,2^8,2^9,2^{10}$ for $\gamma=1/2$. The positions of their peaks approach the critical point $h_c=1$.

Next we consider the anisotropic case. The numerical results for the RFS as a function of h are shown in Fig. 1, from which we can see that, as the system size increases, the peaks become sharper and sharper, and their positions approach the critical point, $h_c=1$. We adopt the $1/N$ expansion method with CUT that was used extensively by Dusuel and Vidal [46,47], which corresponds to the large-N limit. The Holstein-Primakoff method is not suitable for our task, since it could only give a first-order correction in a $1/N$ expansion.

Here we briefly review the CUT introduced by Wegner [42] and independently by Glazek and Wilson [43,44]. For a pedagogical introduction to this technique, one can see [49]. The main idea of the CUT is to diagonalize the Hamiltonian in a continuous way starting from the original Hamiltonian $H=H(l=0)$. A flowing Hamiltonian is then defined by

$$H(l)=U^{\dagger}(l)H(0)U(l), \tag{33}$$

where $U(l)$ is unitary and l is a scaling parameter. A derivation of Eq. (33) with respect to l yields the so-called flow equation

$$\partial_l H(l)=[\eta(l),H(l)], \tag{34}$$

where $\eta(l)=-U(l)^{\dagger}\partial_l U(l)$ is an anti-Hermitian generator. The crucial point is to choose the generator $\eta(l)$ such that $H(\infty)$ is diagonal in the original basis in which $H(0)$ is nondiagonal. The choice of the generator is not unique. Wegner proposed to take the commutator between the diagonal part $H_d(l)$ and the nondiagonal part $H_{nd}(l)$; then the generator reads $\eta_W(l)=[H_d(l),H_{nd}(l)]$. Another possibility is the so-called quasiparticle-conserving generator proposed by Mielke [50] and Knetter and Uhrig [51]. If Q is the operator counting the number of elementary excitations, the matrix elements of $\eta(l)$ in the eigenbasis of Q are chosen to be

$$\eta_{i,j}(l)=\text{sgn}[q_i(l)-q_j(l)]H_{i,j}(l), \tag{35}$$

where $q_i(l)$ is the eigenvalue of $Q(l)$ and $\text{sgn}(x)$ gives the sign of x. Meanwhile, a Hermitian observable $\Omega(l)=U^{\dagger}(l)\Omega(0)U(l)$ is subject to the same flow equation as $H(l)$. Then we can compute the expectation value of $\Omega(0)$ on an eigenstate $|\phi\rangle$ of $H(0)$ as $\langle\phi|\Omega|\phi\rangle=\langle\phi|U(l=\infty)\Omega(l=\infty)U^{\dagger}(l=\infty)|\phi\rangle$, where $U^{\dagger}(l=\infty)|\phi\rangle$ is the eigenstate of $H(l=\infty)$. For detailed calculation, one can see [46,47], in which the asymptotic forms and the scaling exponents of the spin expectation values, $\langle S_z\rangle/N$ and $\langle S_\alpha^2\rangle/N^2$ ($\alpha=x,y,z$), were derived. In the following, we adopt their results of the spin expectation values, and calculate the scaling exponents of their derivatives.

We consider the system size N to be very large, and the matrix elements are rewritten

$$v_\pm = \frac{1}{4} + \frac{\langle S_z^2\rangle}{N^2} \pm \frac{\langle S_z\rangle}{N},$$

$$y = \frac{1}{4} - \frac{\langle S_z^2\rangle}{N^2}, \quad u = \frac{\langle S_x^2\rangle - \langle S_y^2\rangle}{N^2}. \tag{36}$$

The spin expectation values appeared in the above expressions can be solved by the CUT with $1/N$ expansion. For the symmetry phase ($h>1$), we have

$$\frac{2\langle S_z\rangle}{N} = 1 + \frac{1}{N}\left(\frac{P_z^{(1)}}{G^{1/2}}+1\right) + \frac{(1-\gamma)^2}{N^2}\left(\frac{P_z^{(2)}}{G^2}+\frac{Q_z^{(2)}}{G^{3/2}}\right) + \frac{(1-\gamma)^2}{N^3}\left(\frac{P_z^{(3)}}{G^{7/2}}+\frac{Q_z^{(3)}}{G^3}\right) + O\left(\frac{1}{N^4}\right),$$

$$\frac{4\langle S_x^2\rangle}{N^2} = (h-\gamma)\left[\frac{1}{NG^{1/2}} + \frac{1}{N^2}\left(\frac{P_{xx}^{(2)}}{G^2}+\frac{Q_{xx}^{(2)}}{G^{3/2}}\right) + \frac{1}{N^3}\left(\frac{P_{xx}^{(3)}}{G^{7/2}} + \frac{Q_{xx}^{(3)}}{G^3}\right)\right] + O\left(\frac{1}{N^4}\right),$$

$$\frac{4\langle S_y^2\rangle}{N^2} = \frac{1}{h-\gamma}\left[\frac{G^{1/2}}{N} + \frac{1}{N^2}\left(\frac{P_{yy}^{(2)}}{G}+\frac{Q_{yy}^{(2)}}{G^{1/2}}\right) + \frac{1}{N^3}\left(\frac{P_{yy}^{(3)}}{G^{5/2}} + \frac{Q_{yy}^{(3)}}{G^2}\right)\right] + O\left(\frac{1}{N^4}\right),$$

$$\frac{4\langle S_z^2\rangle}{N^2} = 1 + \frac{1}{N}\left(\frac{P_{zz}^{(1)}}{G^{1/2}}+2\right) + \frac{1}{N^2}\left(\frac{P_{zz}^{(2)}}{G^2}+\frac{Q_{zz}^{(2)}}{G^{3/2}}\right) + \frac{(1-\gamma)^2}{N^3}\left(\frac{P_{zz}^{(3)}}{G^{7/2}}+\frac{Q_{zz}^{(3)}}{G^3}\right) + O\left(\frac{1}{N^4}\right), \tag{37}$$

where $G\equiv G(h,\gamma)=(h-1)(h-\gamma)$. Here, $P_\xi^{(i)}\equiv P_\xi^{(i)}(h,\gamma)$ and $Q_\xi^{(i)}\equiv Q_\xi^{(i)}(h,\gamma)$ ($i=1,2,3$ and $\xi=z,xx,yy,zz$) are complicated polynomials of h and γ; for more details, one can refer to the Appendix of [47]. We note that the above spin expectation values, denoted by Φ, can be written in the form

$$\Phi_N(h,\gamma) = \Phi_N^{\text{reg}}(h,\gamma) + \Phi_N^{\text{sing}}(h,\gamma), \tag{38}$$

where the superscripts "reg" and "sing" stand for regular and singular, respectively. The regular part is understood to be a function of h, which is nonsingular at $h=1$, as well as all its derivatives. Take $2\langle S_z\rangle/N$ for example; the regular part is $1+1/N$ and the remainder forms the singular part. As h approaches 1, the terms involving $Q_\xi^{(i)}$ are small compared to the terms involving $P_\xi^{(i)}$ by a factor $G(h,\gamma)$. Therefore, we consider only the terms involving $P_\xi^{(i)}$.

Now we show how to compute the scaling exponents of the spin expectation values, the method used by Vidal *et al.* [47]. Take $2\langle S_z\rangle/N$ for example,

$$\frac{2\langle S_z\rangle}{N} = 1 + \frac{1}{N} + \frac{1}{NG^{1/2}}\left[P_z^{(1)} + \frac{(1-\gamma)^2P_z^{(2)}}{NG^{3/2}} + \frac{(1-\gamma)^2P_z^{(3)}}{(NG^{3/2})^2} + O\left(\frac{1}{(NG^{3/2})^3}\right)\right], \tag{39}$$

where the singular part, terms after $1+1/N$, can be written in the form

$$\left(\frac{2\langle S_z\rangle}{N}\right)^{\text{sing}} \simeq \frac{1}{NG(h,\gamma)^{1/2}}\mathcal{F}_{S_z}[NG(h,\gamma)^{3/2},\gamma], \tag{40}$$

where $\mathcal{F}_\Phi$ ($\Phi=S_z,S_x^2,S_y^2,S_z^2$) is a scaling function for these spin expectation values. In fact, there can be no singularity in any physical quantity in a finite-size system, and the critical point $h_c=1$ only for the thermodynamic limit $N\to\infty$. This implies that the singularity of $G(h,\gamma)^{-1/2}$ has to be canceled by that of $\mathcal{F}_{S_z}[NG(h,\gamma)^{3/2},\gamma]$. Thus one must have $\mathcal{F}_{S_z}(x,\gamma)\sim x^{1/3}$, which in turn implies the following finite-size scaling:

$$\left.\frac{2\langle S_z\rangle}{N}\right|_{h=1} \sim \frac{a_z^{(0)}}{N^{2/3}}. \tag{41}$$

Immediately, one can obtain the asymptotic forms of all the spin expectation values:

$$\left.\frac{2\langle S_z\rangle}{N}\right|_{h=1} \sim 1 + \frac{1}{N} + \frac{a_z^{(0)}}{N^{2/3}},$$

$$\left.\frac{4\langle S_x^2\rangle}{N^2}\right|_{h=1} \sim \frac{a_{xx}^{(0)}}{N^{2/3}},$$

$$\left.\frac{4\langle S_y^2\rangle}{N^2}\right|_{h=1} \sim \frac{a_{yy}^{(0)}}{N^{4/3}},$$

$$\left.\frac{4\langle S_z^2\rangle}{N^2}\right|_{h=1} \sim 1 + \frac{2}{N} + \frac{a_{zz}^{(0)}}{N^{2/3}}, \tag{42}$$

where $a_\xi^{(0)}$ ($\xi=z,xx,yy,zz$) are all constants depending on γ. Then take the first-order derivatives of Eq. (37) with respect to h; one can find similar scaling functions with Eq. (40). Here we also consider $2\langle S_z\rangle/N$,

$$\left(\frac{\partial}{\partial h}\frac{2\langle S_z\rangle}{N}\right)^{\text{sing}} \simeq \frac{1}{NG(h,\gamma)^{3/2}}\mathcal{G}_{S_z}[NG(h,\gamma)^{3/2},\gamma], \tag{43}$$

where $\mathcal{G}_\Phi$ is a scaling function for the derivatives of spin expectation values, and then we find the finite-size scaling

$$\left.\frac{\partial}{\partial h}\frac{2\langle S_z\rangle}{N}\right|_{h=1} \sim a_z^{(1)}. \tag{44}$$

The scaling forms of other derivatives are

$$\left.\frac{\partial}{\partial h}\frac{4\langle S_x^2\rangle}{N^2}\right|_{h=1} \sim a_{xx}^{(1)},$$

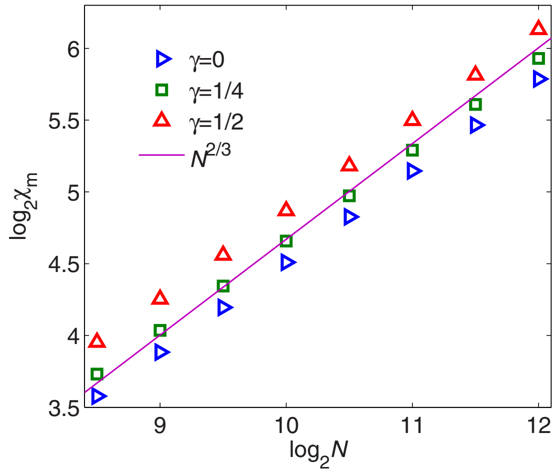

FIG. 2. (Color online) Maximum susceptibility χ_m as a function of system size N. We can see that the numerical results approach the solid line with slope 2/3 as the system size increases.

$$\frac{\partial}{\partial h}\frac{4\langle S_y^2\rangle}{N^2}\bigg|_{h=1} \sim \frac{a_{yy}^{(1)}}{N^{2/3}},$$

$$\frac{\partial}{\partial h}\frac{4\langle S_z^2\rangle}{N^2}\bigg|_{h=1} \sim a_{zz}^{(1)}, \tag{45}$$

where $a_{\xi}^{(1)}$ ($\xi=z,xx,yy,zz$) are constants depending on γ. As we can see, except for $4\langle S_y^2\rangle/N^2$, the other first-order derivatives are all independent of N. With the help of Eq. (24), we find that the maximum RFS $\chi_m\equiv\chi(h_m,N,\gamma)$ is

$$\chi_m \sim -\frac{(a_{zz}^{(1)})^2 N}{a_{zz}^{(0)}N^{1/3}+2} \tag{46}$$

for large N. Here we just present the divergent term. It is noticeable that $a_{zz}^{(0)}$ should be less than -2 to ensure the matrix element $y>0$; thus $\chi_m>0$. Then we have

$$\ln \chi_m = A_N \ln N + \text{const}, \tag{47}$$

where the constant depends only on γ and the scaling exponent A_N approaches 2/3 as N increases, which is verified numerically, as shown in Fig. 2. In the broken-symmetry phase ($0\leqslant h<1$), with a similar procedure, we can derive the same exponent. However, for global fidelity susceptibility, the scaling exponent is 1.33 [24].

The difference of the scaling exponent between global and reduced fidelity susceptibilities is very interesting. We denote the global and the reduced fidelities as F_G and F_R. It is known that $F_G\leqslant F_R$ [1]; thus the corresponding susceptibilities satisfy $\chi_G\geqslant\chi_R$, according to Eq. (4). If the finite-size scaling behaviors are $\chi_G\sim N^{\alpha}$ and $\chi_R\sim N^{\beta}$, one readily knows that the scaling exponents satisfy $\beta\leqslant\alpha$. Here we consider a simple case, an N-body system represented by a product state that reads

$$|\psi(h)\rangle = \bigotimes_{i=1}^{N}|\phi_i(h)\rangle; \tag{48}$$

each subsystem is in a pure state. We denote the one-body reduced fidelity as F_i, the relation between the global and the reduced fidelities is

$$\begin{aligned} F_G(h,\delta) &= |\langle\psi(h)|\psi(h+\delta)\rangle| \\ &= \prod_{i=1}^{n}|\langle\phi_i(h)|\phi_i(h+\delta)\rangle| \\ &= \prod_{i=1}^{n}F_i(h,\delta). \end{aligned} \tag{49}$$

According to Eq. (4), we have $\chi_G=\Sigma_{i=1}^{N}\chi_R$; moreover, if the system has translational symmetry, $\chi_G=N\chi_R$. However, if the global state is entangled, i.e., there exist interactions between particles, it is not easy to give a quantitative relation between F_G and F_R.

IV. CONCLUSION

In summary, we have investigated the RFS for a second-order quantum phase transition of the LMG model. For the case that ρ is block diagonal in 2×2 matrices, we derive a general formula for the RFS. By using a mean-field approximation, we obtain the critical behavior of the RFS for all γ in the thermodynamic limit. Then with the CUT the finite-size scaling exponent of the RFS is obtained analytically and confirmed numerically. Our results show that the RFS undergoes singularity around the critical point, indicating that the RFS can be used to characterize the QPTs. It is suggested that we can extract information about the QPTs from only the fidelity of a subsystem, without probing the global system, which is of practical significance in experiments.

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